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STANDARD ERRORS FOR OBLIQUELY ROTATED FACTOR LOADINGS

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Abstract

In a manner similar to that used in the orthogonal case, formulas for the asymptotic standard errors of analytically rotated oblique factor loading estimates are obtained. This is done by finding expressions for the partial derivatives of an oblique rotation algorithm and using previously derived results for unrotated loadings. These include the results of Lawley for maximum likelihood factor analysis and those of Girshick for principal components analysis. Details are given in cases including direct oblimin and direct Crawford-Ferguson rotation. Numerical results for an example involving maximum likelihood estimation with direct quartimin rotation are presented. They include simultaneous tests for significant loading estimates.

STANDARD ERRORS FOR OBLIQUELY ROTATED FACTOR LOADINGS

1. Introduction

In an earlier paper Archer and Jennrich [1973] derived formulas for the asymptotic standard errors of orthogonally rotated factor loading estimates. Corresponding results are derived here for obliquely rotated loadings.

We begin with a p by k factor loading matrix $A = (\alpha_{ir})$ and an asymptotically normal estimate $\hat{A} = (\hat{\alpha}_{ir})$. Maximum likelihood estimates in the classical factor analysis model [Lawley, 1967] and the principal components estimates in principal components analysis [Girshick, 1939] are both of this form. We are interested in the effect of an oblique rotation algorithm h on the asymptotic distribution of $\hat{A}$. A function h is an oblique rotation algorithm if it maps an arbitrary p by k matrix X into a p by k matrix $Y = XT$ where T is a nonsingular k by k matrix whose inverse has normalized rows, i.e.,

$$\operatorname{diag}(T'T)^{-1} = I \quad ,$$

the k by k identity matrix. The value of T may, and generally will, be a function of X . We are interested in the asymptotic distribution of the rotated loading estimates $\hat{\Lambda} = h(\hat{A}) = \hat{A}\hat{T}$ and the rotated factor correlation estimates $\hat{\Phi} = (\hat{T}'\hat{T})^{-1}$. Of special interest are the cases when h represents oblimin [Harman, 1967, p. 324] or Crawford-Ferguson [1970] rotation.

2. Asymptotic Distributions under Oblique Rotation

Let $\Lambda = h(A) = (\lambda_{ij})$ denote the "true" rotated loadings and $\Phi = g(A) = (T'T)^{-1} = (\phi_{uv})$ the true rotated factor correlations. Assume that at A, h has a differential dh and g has a differential dg. Then

$$(1) \qquad \sqrt{n}\,(\hat{\Lambda} - \Lambda) \overset{a}{=} dh(\sqrt{n}\,(\hat{A} - A))$$

and

$$(2) \qquad \sqrt{n}\,(\hat{\Phi} - \Phi) \overset{a}{=} dg(\sqrt{n}\,(\hat{A} - A))$$

where " $\overset{a}{=}$ " means that the difference between the left- and right-hand sides of (1) and (2) approaches zero in probability as the sample size n on which the estimate $\hat{A}$ is based approaches infinity [Rao, 1965, p. 321]. Since dh and dg are linear and $\hat{A}$ is an asymptotically normal estimate of A, $\hat{\Lambda}$ and $\hat{\Phi}$ are asymptotically normal estimates of Λ and Φ whose asymptotic covariance matrices may be obtained from that of $\hat{A}$. In terms of the partial derivatives of h and g,

$$(3) \qquad \operatorname{acov}(\hat{\lambda}_{ir}, \hat{\lambda}_{js}) = \Sigma_{mnxy} \frac{\partial h_{ir}}{\partial \alpha_{mx}} \operatorname{acov}(\hat{\alpha}_{mx}, \hat{\alpha}_{ny}) \frac{\partial h_{js}}{\partial \alpha_{ny}}$$

and

$$(4) \qquad \operatorname{acov}(\hat{\phi}_{rs}, \hat{\phi}_{uv}) = \Sigma_{mnxy} \frac{\partial g_{rs}}{\partial \alpha_{mx}} \operatorname{acov}(\hat{\alpha}_{mx}, \hat{\alpha}_{ny}) \frac{\partial g_{uv}}{\partial \alpha_{ny}} \quad .$$

Formulas for the asymptotic covariances on the right, i.e., the asymptotic covariances for specific unrotated loadings, were given by Lawley [1967] for maximum likelihood factor analysis and by Girshick [1939] for principal components analysis. (See Lawley and Maxwell [1971] for a convenient summary.)

For the oblique rotation algorithms used in practice, it is difficult to find dh and dg or, equivalently, the partial derivatives $\partial h_{ir}/\partial \alpha_{js}$ and $\partial g_{uv}/\partial \alpha_{js}$ directly. As in the orthogonal case [Archer & Jennrich, 1973] we proceed by means of implicit differentiation. Suppose that

$$(5) \qquad \psi(\Lambda,\Phi) = 0$$

is a $k(k-1)$ dimensional constraint which is satisifed whenever $\Lambda = AT$ is an h -rotation of A and $\Phi = (T'T)^{-1}$ is the corresponding factor correlation matrix. Constraints of this form will be derived in Section 3. While it would be possible to express these constraints in terms of A and T rather than in terms of Λ and Φ, the latter parameters appear to be the more natural and lead to simpler formulas. It is not possible here, as it was in the orthogonal case, to express the constraints in terms of Λ alone. In the development of the general theory, this represents the primary difference from the orthogonal case. Differentiating the relations:

$$(6) \qquad \Lambda = AT$$

$$(7) \qquad \Phi = (T'T)^{-1}$$

(8) $\quad \operatorname{diag} \Phi = I$

(9) $\quad \psi(\Lambda,\Phi) = 0$

gives:

(10) $\quad d\Lambda = dAT + AdT$

(11) $\quad d\Phi = -\Phi(T'dT + dT'T)\Phi$

(12) $\quad \operatorname{diag} d\Phi = 0$

(13) $\quad d\psi_1(d\Lambda) + d\psi_2(\Phi) = 0$

when $d\psi_1$ and $d\psi_2$ denote the differentials at (Λ,Φ) of ψ with respect to its first and second arguments respectively. Solving (10) through (13) for $d\Lambda$ in terms of dA will define the differential dh and the required partial derivatives of h. Similarly solving for $d\Phi$ in terms of dA will define dg and the required partial derivatives of g. This is the standard technique of implicit differentiation. The only novelty here is that some matrix algebra is employed. Let

(14) $\quad K = T^{-1}dT\Phi \quad$ so $\quad dT = TK\Phi^{-1}$.

Using this change of variable, the linearity of $d\psi_1$, and simplifying slightly, equations (10) through (13) become

(15) $\quad d\Lambda = dAT + \Lambda K\Phi^{-1}$

(16) $d\Phi = -(K + K')$

(17) $\operatorname{diag} K = 0$

(18) $d\psi_1(dAT) + d\psi_1(\Lambda K\Phi^{-1}) - d\psi_2(K + K') = 0$.

We shall solve the last equation for K in terms of dA . To this end, let $\mathcal{N}$ be the space of k by k matrices which are zero on the diagonal and define the linear transformation

(19) $L(X) = d\psi_1(\Lambda X\Phi^{-1}) - d\psi_2(X + X')$

for all X in $\mathcal{N}$. Because of (17), K is in $\mathcal{N}$. Assuming L is an invertible linear transformation, (18) gives

(20) $K = -L^{-1}(d\psi_1(dAT))$.

Substituting this expression into (15) gives

(21) $d\Lambda = dAT - \Lambda L^{-1}(d\psi_1(dAT))\Phi^{-1}$

which expresses $d\Lambda$ in terms of dA and defines the differential dh . Similarly substituting (20) into (16) gives

(22) $d\Phi = -L^{-1}(d\psi_1(dAT)) - (L^{-1}(d\psi_1(dAT)))'$

and this defines dg . Equations (21) and (22) represent our basic results. The approach in the following sections will be to find constraint functions ψ suitable for various types of oblique rotation and then to recover the required partial derivatives of h and g from (21) and (22) respectively.

3. Constraints for an Oblique Rotation Algorithm

Oblique rotation may be applied directly to the loadings [Jennrich & Sampson, 1966] or indirectly to the reference factor structure [Carroll, 1960]. This gives rise to direct and indirect oblique rotation methods (see Harman [1967, p. 334] for a discussion). Because the present trend seems to be toward direct methods, we shall consider only that case. No corresponding decision was required when considering orthogonal rotation for there the loadings and reference structure are the same.

Direct oblique analytic rotation algorithms are designed to optimize a criterion of the form:

$$(23) \qquad Q = Q(\Lambda) = Q(AT)$$

over all T which satisfy (7) and (8). The resulting $\Lambda = AT$ is called a Q-rotation of A. It is not necessary that the optimizing Λ be unique. It is assumed only that the corresponding rotation algorithm h maps each A into an optimizing Λ. The criterion Q is usually a quartic function of the components of Λ but it may, as in the case of target rotation, be a quadratic function. Since there is no need to specialize at this point, an arbitrary Q will be considered here and in the next section.

Let dQ denote the differential of Q at Λ and assume that for a given A, $\Lambda = AT$ optimizes Q for all T satisfying (7) and (8). Then it is necessary that

(24) $dQ(AdT) = 0$

for all dT satisfying (11) and (12). But using the change of variable in (14), this is equivalent to requiring that

(25) $dQ(\Lambda K \Phi^{-1}) = 0$

for all K in $\mathcal{M}$. Let $K(u,v)$ be the elementary k by k matrix which has a one in row u and column v and is zero elsewhere. Then (25) is equivalent to requiring

(26) $\psi_{uv} = dQ(\Lambda K(u,v)\Phi^{-1}) = 0$

for $1 \leq u \neq v \leq k$. These are constraints of the form (5) required in Section 2. In coordinate form the constraint functions are:

(27) $$\psi_{uv} = \sum_{i=1}^{p} \sum_{r=1}^{k} \frac{\partial Q}{\partial \lambda_{ir}} \lambda_{iu} \phi^{ur}$$

for $1 \leq u \neq v \leq k$. In matrix form the constraints require that

(28) $\Lambda' \frac{dQ}{d\Lambda} \Phi^{-1}$ be symmetric.

Here $dQ/d\Lambda$ denotes the matrix $(\partial Q/\partial \lambda_{ir})$ of partial derivatives of Q. Except for the appearance of Φ^{-1} these are the same as the constraints which arose in the orthogonal case [Archer & Jennrich, 1973].

4. The Partial Derivatives of the Functions h and g

We shall derive here explicit and computationally convenient formulas for the partial derivatives of h and g. To this end let $J(j,s)$ be the elementary p by k matrix which has a one in row j and column s and is zero elsewhere. In terms of their differentials dh and dg, the partial derivatives of g and h are:

$$(29) \qquad \frac{\partial h_{ir}}{\partial \alpha_{js}} = dh_{ir}(J(j,s))$$

and

$$(30) \qquad \frac{\partial g_{uv}}{\partial \alpha_{js}} = dg_{uv}(J(j,s))$$

for $1 \leq i,j \leq p$ and $1 \leq r,s,u,v \leq k$. Another way to express (29) is to assert that

$$(31) \qquad H = \left(\frac{\partial h_{ir}}{\partial \alpha_{js}}\right)$$

is the matrix of dh relative to the basis

$$(32) \qquad J(j,s) \quad , \quad 1 \leq j \leq p \quad , \quad 1 \leq s \leq k$$

of the space $\mathcal{M}$ of all p by k matrices. Similarly

$$(33) \qquad G = \left(\frac{\partial g_{uv}}{\partial \alpha_{js}}\right)$$

is the matrix of dg relative to this basis. The differential dh is defined by the fundamental relation (21). We may view (21) as a composition of four linear transformations and express the matrix H of the linear transformation dh in the form

$$(34) \qquad H = B - CD^{-1}E$$

where B , C , D , and E are the matrices of the linear transformations

$$(35) \qquad X \mapsto XT \quad , \qquad X \in \mathcal{M}$$

$$(36) \qquad Z \mapsto \Lambda Z \Phi^{-1} \quad , \qquad Z \in \mathcal{N}$$

$$(37) \qquad Z \mapsto L(Z)' \quad , \qquad Z \in \mathcal{N}$$

$$(38) \qquad X \mapsto d\psi_1(XT) \quad , \qquad X \in \mathcal{M}$$

respectively relative to the basis (32) of $\mathcal{M}$ and the basis

$$(39) \qquad K(u,v) \quad , \qquad 1 \leq u \neq v \leq k$$

of $\mathcal{N}$. Similarly since dg is defined by the fundamental relation (22), the matrix G of the linear transformation dg has the form

$$(40) \qquad G = -(F + F')$$

where

$$(41) \qquad F = D^{-1}E \quad .$$

From (35) the matrix B is given by:

$$(42) \qquad B_{ir,js} = (J(j,s)T)_{ir} = \delta_{ij}T_{sr}$$

where $(\ \cdot\ )_{ir}$ denotes the element in row i and column r of the matrix inside the parentheses and δ_{ij} denotes the Kronecker delta.

Similarly from (36) the matrix C is given by:

$$(43)\qquad C_{ir,uv} = (\Lambda K(u,v)\Phi^{-1})_{ir} = \lambda_{iu}\phi^{vr} \quad .$$

Since D is the matrix of the linear transformation L defined by (19) it is given by:

$$(44)\qquad D_{uv,xy} = [d\psi_1(\Lambda K(x,y)\Phi^{-1}) - d\psi_2(K(x,y) + K(y,x))]_{uv}$$

$$= \sum_{i=1}^{p} \sum_{r=1}^{k} \frac{\partial\psi_{uv}}{\partial\lambda_{ir}} \lambda_{ix}\phi^{yr} - \frac{\partial\psi_{uv}}{\partial\phi_{xy}} - \frac{\partial\psi_{uv}}{\partial\phi_{yx}} \quad .$$

Finally using (38), E is given by:

$$(45)\qquad E_{xy,js} = (d\psi_1(J(j,s)T))_{xy}$$

$$= \sum_{r=1}^{k} \frac{\partial\psi_{xy}}{\partial\lambda_{jr}} T_{sr} \quad .$$

In each case the index ranges are $1 \leq i,j \leq p$, $1 \leq r,s \leq k$, $1 \leq u \neq v \leq k$, and $1 \leq x \neq y \leq k$.

Equations (44) and (45) require partial derivatives of the constraint functions ψ_{uv} . As discussed in the previous section these depend on the rotation algorithm employed. Specific formulas for the generalized Crawford-Ferguson family will be obtained in the next section.

5. The Generalized Crawford-Ferguson Family

There are two well-known families of fourth degree oblique rotation criteria, the oblimin criteria [Harman, 1967, p. 324]

$$(46) \qquad \sum_{r\neq s} \left(\sum_{i=1}^{p} \lambda_{ir}^2 \lambda_{is}^2 - \frac{\gamma}{p} \sum_{i=1}^{p} \lambda_{ir}^2 \sum_{i=1}^{p} \lambda_{is}^2 \right)$$

and the Crawford-Ferguson [1970] criteria:

$$(47) \qquad K_1 \sum_{r\neq s} \sum_{i=1}^{p} \lambda_{ir}^2 \lambda_{is}^2 + K_2 \sum_{i\neq j} \sum_{r=1}^{k} \lambda_{ir}^2 \lambda_{jr}^2 \quad .$$

Specific criteria in these families are obtained by fixing the values of γ and K_1 and K_2. In spite of the fact that the Crawford-Ferguson family contains an additional parameter, it does not contain the oblimin family. Both families, however, are contained in what we shall call the generalized Crawford-Ferguson family:

$$(48) \qquad \kappa_1 \left(\sum_{i=1}^{p} \sum_{r=1}^{k} \lambda_{ir}^2 \right)^2 + \kappa_2 \sum_{i=1}^{p} \left(\sum_{r=1}^{k} \lambda_{ir}^2 \right)^2 + \kappa_3 \sum_{r=1}^{k} \left(\sum_{i=1}^{p} \lambda_{ir}^2 \right)^2 + \kappa_4 \sum_{i=1}^{p} \sum_{r=1}^{k} \lambda_{ir}^4 \quad .$$

This family becomes the Crawford-Ferguson family when $\kappa_1 = K_1 + K_2$, $\kappa_2 = -K_2$, $\kappa_3 = -K_1$, and $\kappa_4 = 0$ and it becomes the oblimin family when $\kappa_1 = -\gamma/p$, $\kappa_2 = 1$, $\kappa_3 = \gamma/p$, and $\kappa_4 = -1$. Indeed, every fourth degree polynomial criterion which has the property that it is invariant under changes in sign of the rows and columns of Λ and under row and

column permutations is of the form given in (48). We will derive our results for this complete family of quartic criteria. Let $4Q$ equal the expression given in (48). It follows easily that

$$(49) \qquad \frac{\partial Q}{\partial \lambda_{ir}} = \lambda_{ir} M_{ir}$$

where

$$(50) \qquad M_{ir} = \kappa_1 \Sigma_i \Sigma_r \lambda_{ir}^2 + \kappa_2 \Sigma_r \lambda_{ir}^2 + \kappa_3 \Sigma_i \lambda_{ir}^2 + \kappa_4 \lambda_{ir}^2$$

for $1 \leq i \leq p$ and $1 \leq r \leq k$. Thus using (27),

$$(51) \qquad \psi_{uv} = \sum_{i=1}^{p} \sum_{r=1}^{k} \lambda_{iu} \lambda_{ir} M_{ir} \phi^{vr} \quad , \quad 1 \leq u \neq v \leq k$$

are the appropriate constraint functions for the generalized Crawford-Ferguson family. One may show easily that

$$(52) \qquad \frac{\partial M_{ir}}{\partial \lambda_{js}} = 2(\kappa_1 + \kappa_2 \delta_{ij} + \kappa_3 \delta_{rs} + \kappa_4 \delta_{ij} \delta_{rs}) \lambda_{js}$$

and tnen with a little more straightforward differentiation effort that

$$(53) \qquad \frac{\partial \psi_{uv}}{\partial \lambda_{ir}} = \delta_{ur} \left(\frac{dQ}{d\Lambda} \Phi^{-1} \right)_{iv} + M_{ir} \lambda_{iu} \phi^{rv}$$

$$+ 2\kappa_1 \lambda_{ir} (\Lambda' \Lambda \Phi^{-1})_{uv} + 2\kappa_2 \lambda_{ir} \lambda_{iu} (\Lambda \Phi^{-1})_{iv}$$

$$+ 2\kappa_3 \lambda_{ir} (\Lambda' \Lambda)_{ur} \phi^{rv} + 2\kappa_4 \lambda_{ir}^2 \lambda_{iu} \phi^{rv}$$

for $1 \leq u \neq v \leq k$, $1 \leq i \leq p$, and $1 \leq r \leq k$. Having completed this task it is quite easy to show that

$$(54) \qquad \frac{\partial \psi_{uv}}{\partial \phi_{xy}} = -(\delta_{ux}\phi^{yv} + \delta_{vy}\phi^{xv})(\Lambda' \frac{dQ}{d\Lambda} \phi^{-1})_{uu}$$

for $1 \leq u \neq v \leq k$ and $1 \leq x \neq y \leq k$. These are the partial derivatives of ψ required in Section 4.

6. Example and Discussion

As in previous work on the orthogonal case, the results derived here apply equally well to principal components analysis and to maximum likelihood factor analysis. Because of the work of Jöreskog [1967], Jennrich and Robinson [1969], and Clarke [1970] maximum likelihood factor analysis has become computationally feasible. While this may enhance the popularity of maximum likelihood factor analysis, principal components analysis is too important and certainly too popular to be ignored.

We shall consider a maximum likelihood example with direct quartimin rotation. Table 1 contains unrotated loading estimates obtained by Jöreskog [1967] from an analysis of the correlations of 9 variables measured on 211 subjects. These data were originally analyzed by Emmett

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Insert Table 1 about here

[1949]. It was chosen here because Lawley and Maxwell [1971, p. 64] have given standard errors for the loadings in Table 1 using Lawley's [1967] formulas. We have verified these results which indicates that in

all probability Lawley's formulas have been properly implemented both here and by Lawley and Maxwell. Since these formulas are not simple, this is worth verifying. The verification was exact to within one digit in the last decimal place presented in Lawley and Maxwell's text.

Table 2 contains a direct quartimin rotation of the loadings in Table 1 together with the matrix T which produced the transformation. This is a fairly clean rotation. Variables 4, 5, and 6 appear to be primarily associated with factor 2, variable 8 with factor 3, and the remaining with factor 1. But how stable are the factor loading estimates? Table 3 contains standard errors computed from the formulas derived here together with those of Lawley.* The standard errors are pleasantly, almost surprisingly, small. They range from .036 to .141. This is similar to what happened in the orthogonal case [Archer & Jennrich, 1973] using a different set of data but one with roughly the same number of variables and sample size. To our knowledge this is the first time that standard errors for obliquely rotated factor loading estimates have been published.

Insert Tables 2 and 3 about here

The standard errors presented give a quick indication of the stability of the rotated loading estimates. For example one may use the standard errors in Table 3 to scan Table 2 for loadings which are significantly different from zero while accounting for the fact that he is scanning. To this end we observe that asymptotically at least

*There has been a recent indication that the Lawley [1967] formulas may have a minor error. This does not affect the theory developed here, but it could affect the values in this example.

$$\frac{|\hat{\lambda}_{ir} - \lambda_{ir}|}{\text{std } \hat{\lambda}_{ir}} \leq \chi_{.95}(k(p - k + 1)) \quad , \quad 1 \leq i \leq p \, , \quad 1 \leq r \leq k \tag{55}$$

is a 95% simultaneous confidence interval for all pk loadings. Here $\chi_{.95}(\nu)$ denotes the square root of the 95% probability point of the chi-squared distribution with ν degrees of freedom. In our example $\nu = k(p - k + 1) = 21$ so

$$\chi_{.95}(21) = \sqrt{32.7} = 5.72 \quad . \tag{56}$$

The entries in Table 2 which after division by the corresponding entries in Table 3 exceed this value have been marked with an asterisk. They are the estimates which are significantly different from zero at the 95% probability level. It is interesting to note that since the median standard error in Table 3 is .067, the median significant deviation from zero is .383. This is not too far from the value .30 suggested by the "rule of thirty" and often found in factor analytic studies. As can be seen by multiplying each entry in Table 3 by 5.72 the rule of thirty represents a fairly gross approximation but not one which is completely wide of the mark. Standard errors depend, of course, on sample size. For this reason the rule of thirty is often used with the understanding that, as here, there are about ten observations per loading.

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TABLE 1

Unrotated Standardized Maximum Likelihood Loadings

Variate	Factor I	Factor II	Factor III	Communality
1	.664	.321	.074	.550
2	.689	.247	-.193	.573
3	.493	.302	-.222	.383
4	.837	-.292	-.035	.788
5	.705	-.315	-.153	.619
6	.819	-.377	.105	.823
7	.661	.396	-.078	.600
8	.458	.296	.491	.538
9	.766	.427	-.012	.769

TABLE 2

Direct Quartimin Rotation of the Loadings in Table 1

	Factor			
Variate	I	II	III	Communality
1	.596*	.084	.191	.550
2	.703*	.131	-.104	.573
3	.690*	-.056	-.140	.383
4	.102	.827*	-.023	.788
5	.093	.749*	-.163	.619
6	.087	.932*	.109	.823
7	.772*	-.025	.043	.600
8	.204	.057	.611	.538
9	.811*	.019	.127	.769

Transformation Matrix T

.461	.623	.081
1.053	-1.069	.182
-.650	.180	1.059

TABLE 3

Standard Errors for the Rotated Loadings in Table 2

	Factor		
Variate	I	II	III
1	.074	.082	.096
2	.069	.082	.073
3	.072	.053	.067
4	.054	.084	.075
5	.065	.069	.041
6	.046	.058	.056
7	.064	.036	.141
8	.064	.081	.116
9	.046	.050	.059